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GUT Scalar Potentials for Higgs Inflation

Martin B Einhorn,^{a,1} D R Timothy Jones^b

^aKavli Institute for Theoretical Physics,

University of California, Santa Barbara CA 93106-4030, USA

^bDept. of Mathematical Sciences,

University of Liverpool, Liverpool L69 3BX, UK

E-mail: meinhorn@umich.edu, drtj@liv.ac.uk

Abstract.

Motivated by the idea that there is new physics beyond the Standard Model (SM), we have investigated a number of models for Grand Unified Theories (GUTS) in four dimensions for the possibility that their Higgs fields might be responsible for inflation in the early universe. In addition to models having an intrinsic Planck mass parameter, we have entertained classically scale invariant models in which the Planck scale itself as well as the GUT scale is induced by spontaneous breaking of the gauge symmetry. We found that in non-supersymmetric $SU(5)$ with the usual Higgs in the adjoint representation but with large non-minimal coupling to the curvature, there appear to be several possible flat directions that might lead to inflation. Interestingly, the one of lowest energy is the breaking into $SU(3) \otimes SU(2) \otimes U(1)$ that is suggested by gauge coupling unification. Further, we show that this flat direction is stable against small fluctuations in other directions.

We attempted to extend this to similar supersymmetric GUTS, both global and supergravity, but did not succeed in finding a phenomenologically acceptable model of this type. As is often the case, such models suffered either from a negative vacuum energy or from tachyonic modes. We also considered a variant of an “inverted hierarchy” model in which the GUT scale is set by dimensional transmutation, but were unable to find a phenomenologically acceptable model.

Keywords: inflation, GUTs, supersymmetry and cosmology, Higgs, nonminimal coupling

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¹Emeritus Professor, University of Michigan

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1 Introduction

Quantum field theories in curved backgrounds (or with quantised gravity) in four dimensions incorporating scalar fields may contain terms of the generic form

$$\mathcal{L} \supset \sqrt{g} \xi \phi^2 R \tag{1.1}$$

where ξ is a dimensionless coefficient.

In non-supersymmetric theories, ξ is renormalised; with $\xi = 0$ *not* being a fixed point with respect to the renormalisation group. There has been recent interest [1]–[11] in the possibility that the term of this form involving the Higgs boson which is permitted in the Standard Model might suffice to allow the Higgs to be the inflaton. We shall refer to this general paradigm as the Bezrukov-Shaposhnikov (BS) scenario.

The attraction of this concept is obvious; instead of the ad hoc introduction of a new scalar sector specifically engineered to produce an inflationary era, we have a sector already conceived of for other (excellent) reasons performing the same function. The point is that for field values $\langle\phi\rangle$ such that

$$\xi \langle\phi\rangle^2 \gtrsim M_P^2, \quad (1.2)$$

the scalar potential (in the Jordan frame) is nearly flat and amenable to slow-roll inflation. Crucially, it is possible to entertain the possibility $\xi \gg 1$ without (obviously) violating perturbation theory. There has been vigorous debate over the range in field values (of the inflaton) for which the classical field analysis remains valid as an effective field theory; a particularly detailed analysis appears in Ref. [11] (see also Ref. [12]). An interesting question is whether rendering the theory supersymmetric has any impact here; but we will not address this issue in this paper.

Here we explore the possibility that the inflaton might again be a Higgs, but associated with the spontaneous breaking of a GUT rather than the electroweak gauge group; we have in mind in particular the case of a 24 of $SU(5)$; (a case considered some time ago, in fact, by Salopek et al in Ref. [13]). We also discuss the (even more) radical possibility that Eq. (1.1) is in fact the *only* term linear in the curvature in the action; that is to say, the usual Einstein-Hilbert term $M_P^2 R$ (or its supersymmetric generalisation) is absent, and the classical theory is scale invariant. We will explore the ramifications of this general approach elsewhere¹; noting, however, that, as emphasised in Ref. [13]), there are then difficulties with the interpretation of the field associated with the Higgs flat direction as the inflaton since it decouples from the matter fields and hence does not contribute to reheating.

We also generalise to the supersymmetric case, which was first considered in Ref. [24], and further developed in Refs. [25]–[35]. It turns out to be not possible to implement the idea (a Higgs inflaton) in its basic form in either the MSSM [24], or the NMSSM [25], because the candidate flat directions suffer from tachyonic instabilities. It is possible to circumvent this problem by generalising to a non-minimal Kähler potential; we will return to this issue, but in this paper we prefer to concentrate on the minimal case when the Kähler potential is augmented only by terms characterised by *dimensionless* coupling constants. We discuss the form of the scalar potential first in a general class of supersymmetric models, eventually specialising once again to the case of an adjoint field. We present a particularly simple form for the scalar potential both in general and in the special case of $SU(N)$. We also consider in detail the more complicated example of Witten’s “Inverted Hierarchy” model, which involves two chiral adjoint multiplets. In the supersymmetric cases that we explored, we have typically encountered the same instability problem (alluded to above) associated with the MSSM and the NMSSM, but we have not performed a completely general analysis allowing for extrema with complex field values.

¹For other previous work in this general direction, see for example Ref. [14]–Ref. [23].

2 The adjoint case for non-supersymmetric $SU(5)$

Let us begin more generally with $SU(N)$. For a single hermitian adjoint multiplet, the most general possible quartic scale invariant potential in the Jordan frame is

$$V_J(\Phi) = \lambda_1 \text{Tr} \Phi^4 + \lambda_2 (\text{Tr} \Phi^2)^2, \quad (2.1)$$

where $\Phi = \lambda^A \phi^A / \sqrt{2}$, and the ϕ^A are real. Some results from group theory and our notational conventions are to be found in Appendix A.

If we include a term $\xi \text{Tr} \Phi^2 R$ in the Lagrangian in the Jordan frame, then in the Einstein frame, we have (in the usual case when the Lagrangian also contains an Einstein term $M_P^2 R$) the potential [1]

$$V(\Phi) = \left(\frac{M_P^2}{X} \right)^2 V_J(\phi) \quad (2.2)$$

with $X = M_P^2 + \xi \text{Tr} \Phi^2$. In the *scale invariant* case (when the $M_P^2 R$ term is absent) we have instead $X = \xi \text{Tr} \Phi^2$, where M_P is now an *arbitrary* scale introduced in the course of performing the conformal transformation which connects the two frames.

As we indicated earlier, his model was in fact considered in Ref. [13]; the general analysis of the potential which appears below is, however, new.

Now a hermitian matrix can be rendered real and diagonal by means of a unitary transformation. Since V is invariant under an arbitrary unitary transformation, in order to find the extrema of the potential in the Einstein frame it will suffice to consider the case

$$\Phi = \text{diag}(x_1 \cdots x_i \cdots x_N) \quad \text{and traceless}, \quad (2.3)$$

with x_i real.

Let us first analyse the scale invariant case. We find that the x_i satisfy the equation

$$x^3 - \frac{T_4}{T_2} x - \frac{1}{N} T_3 = 0, \quad (2.4)$$

where $T_m = \text{Tr} \Phi^m$. Of course V is not well defined at $\Phi = 0$. Since Eq. (2.4) is a cubic, there are at most three distinct solutions for x_i . For N even, there is the obvious solution

$$\Phi = \Lambda \text{diag}(1, -1, \cdots 1, -1), \quad (2.5)$$

with

$$V = \frac{M_P^4}{4\xi^2} \left(\lambda_2 + \frac{\lambda_1}{N} \right) \quad (2.6)$$

This flat direction represents, in fact, the minimum of V for all even N .

For odd N the situation is more complicated; let us turn to the $SU(5)$ case. Suppose at least one of the x_i is zero. It follows from Eq. (2.4) that $T_3 = 0$. It is therefore easy to see that the possible solutions for $SU(5)$ are $\Phi = \text{diag}(1, -1, 0, 0, 0)$, $\Phi = \text{diag}(1, -1, 1, -1, 0)$. with results for V of

$$V = \frac{M_P^4}{4\xi^2} \left(\lambda_2 + \frac{\lambda_1}{2} \right) \quad (2.7)$$

and

$$V = \frac{M_P^4}{4\xi^2} \left(\lambda_2 + \frac{\lambda_1}{4} \right) \quad (2.8)$$

respectively.

If none of the x_i are zero then we have the possible forms

$$A : \Phi = \text{diag} (1, 1, 1, z, -3 - z) \quad (2.9)$$

$$B : \Phi = \text{diag} (1, 1, z, z, -2 - 2z). \quad (2.10)$$

Substitution of these forms in Eq. (2.2) and plotting V as a function of z (or seeking consistent solutions for all x_i of Eq. (2.4)) reveals that there are in fact solutions corresponding to

$$A : \Phi = \text{diag} (1, 1, 1, -3/2, -3/2) \quad (2.11)$$

$$B : \Phi = \text{diag} (1, 1, 1, 1, -4) \quad (2.12)$$

with

$$V = \frac{M_P^4}{4\xi^2} \left(\lambda_2 + \frac{7\lambda_1}{30} \right) \quad (2.13)$$

and

$$V = \frac{M_P^4}{4\xi^2} \left(\lambda_2 + \frac{13\lambda_1}{20} \right) \quad (2.14)$$

respectively.

Note that the $SU(3) \otimes SU(2) \otimes U(1)$ solution has the smallest energy. It is in fact stable against all quadratic fluctuations, as shown in Appendix D!

Let us turn now to analyse the potential *with* the Einstein term, $X = M_P^2 + \xi T_2$. Then we find (setting $M_P = 1$) that for an extremum the x_i must satisfy

$$\lambda_1(1 + \xi T_2)x^3 + (\lambda_2 T_2 - \lambda_1 \xi T_4)x - \frac{1}{N} \xi \lambda_1 T_2 T_3 = 0. \quad (2.15)$$

It is easy to see that Eq. (2.15) reduces to Eq. (2.4) in the limit $\xi \rightarrow \infty$. However, from Eq. (2.15) it follows at once that

$$V_J(\Phi) = \lambda_1 T_4 + \lambda_2 T_2^2 = 0. \quad (2.16)$$

Thus in the presence of the Einstein term, the only true extrema of the potential

$$V = \frac{V_J(\Phi)}{(1 + \xi T_2)^2} \quad (2.17)$$

have $V = 0$.

This result is perfectly consistent with an inflationary interpretation; at large ξT_2 , the flat direction of lowest energy is the one corresponding to a $SU(3) \otimes SU(2) \otimes U(1)$ vacuum, with slow-roll towards the true minimum at $\Phi = 0$.

Note: we can understand the extremum of Eq. (2.17) in a simple way as follows. Suppose V as defined in Eq. (2.17) has an extremum $V = \bar{V}$ for $x_i = \bar{x}_i$. Then consider $\bar{V}_\lambda = V(\lambda \bar{x}_i)$. Evidently

$$\bar{V}_\lambda = (1 + \xi T_2)^2 \bar{V} \left[\frac{\lambda^4}{(1 + \lambda^2 \xi T_2)^2} \right], \quad (2.18)$$

and from the fact that the function $y = x^4/(1+ax^2)^2$ has, for $a > 0$, a unique extremum (for finite x) at $x = 0$ the result follows.

We now generalise $V_J(\phi)$ by including a mass term as follows:

$$V_J(\Phi) = -m^2 T_2 + \lambda_1 T_4 + \lambda_2 T_2^2 = 0. \quad (2.19)$$

We assume $m^2 \ll M_P^2$. For field values $\xi T_2 \ll M_P^2$, we can ignore the ξ -term and the minimisation of this potential is an old problem; according to Li [36], for

$$\lambda_2 > 0 \quad \text{and} \quad \lambda_2 > -\frac{7}{30}\lambda_1 \quad (2.20)$$

the minimum corresponds to breaking to $SU(3) \otimes SU(2) \otimes U(1)$, with

$$\langle \Phi \rangle = v_\Phi \text{diag}(2, 2, 2, -3, -3) \quad (2.21)$$

and

$$v_\Phi^2 = \frac{m^2}{60\lambda_2 + 14\lambda_1}. \quad (2.22)$$

Notice that given Eq. (2.20), at least for large scales, we have both $v_\Phi^2 > 0$, and $V > 0$ in Eq. (2.13).

Of course for $|\langle \Phi \rangle| \gg v_\Phi$ one should account for the running of m^2 and $\lambda_{1,2}$ between the two scales; but, given Eq. (2.20), it seems natural that the inflationary era described by Eq. (2.13) would terminate with a transition to the broken vacuum. So if we substitute in the original Jordan frame Lagrangian

$$\mathcal{L} = \frac{1}{2}(M_P^2 + \xi \text{Tr} \Phi^2) R + \frac{1}{2} \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) - V_J(\Phi) \quad (2.23)$$

the form

$$\Phi = \frac{h}{\sqrt{30}} \text{diag}(2, 2, 2, -3, -3) \quad (2.24)$$

we obtain, in the first approximation, precisely the results of [1] for the slow-roll parameters, namely (setting $M_P = 1$)

$$\epsilon = \frac{4}{3\xi^2 h^4}, \quad (2.25)$$

$$N = \frac{3\xi(h^2 - h_{\text{end}}^2)}{4}, \quad (2.26)$$

$$\eta = -\frac{4}{3\xi h^2}. \quad (2.27)$$

with $\epsilon \approx 1$ for $h_{\text{end}} \approx 1.07 M_P / \sqrt{\xi}$. Thus for $\xi \sim 10^6$, inflation can terminate naturally with a transition to the broken vacuum at a scale $h_{\text{end}} \sim 10^{16} \text{GeV}$, while $h_0 \sim 9 M_P / \sqrt{\xi} \sim 10^{17} \text{GeV}$.

Thus the simplest $SU(5)$ model (the original Georgi-Glashow model [37]) is compatible with the BS inflation scenario: a Higgs inflaton. This model is, however, generally regarded as unsatisfactory for reasons both theoretical and experimental; most particularly the increasingly precise limit on the proton lifetime. Note, however, that we may anticipate that our scenario exists in a considerable class of $SU(5)$ models, designed, for example, to alleviate the doublet-triplet splitting problem, or to increase somewhat the unification mass so as to reduce the proton decay rate. For some recent examples see [38]-[41]. Thus it does seem to us that this formulation of Higgs inflation is of interest.

We turn now to the supersymmetric case; more popular than the non-supersymmetric case for reasons which are well known; though not without its own problems, regarding, for example, dimension 5 contributions to proton decay which are of course absent in the non-supersymmetric case. We will find that we are unable to construct a simple model with a stable trajectory of the kind we have identified above.

3 The supersymmetric case

Let us consider the form of the scalar potential in general gauge invariant $N = 1$ supergravity models with chiral supermultiplets; and with a Kähler potential modified (in the Einstein frame) by the inclusion of a term which in the Jordan frame corresponds to adding to the Lagrangian the term

$$\mathcal{L} \supset -6 \int d^2\Theta \mathcal{E} X(\Phi) R + \text{c.c.}, \quad (3.1)$$

where $X(\Phi)$ is *quadratic* in the chiral supermultiplet Φ . Such a term is a natural supersymmetric generalisation of Eq. (1.1). One can of course entertain the possibility that X contains higher powers of Φ ; and indeed these may be used to mitigate the tachyonic instabilities which we will encounter. For a viable model of this type, see for example Ref. [26]. However doing so does call into question the generality of any conclusions reached against contributions from yet other higher dimension operators added to the Kähler potential or superpotential.

We will find that, generally speaking, for theories with or without the $M_P^2 R$ there exist natural flat directions that are candidates for slow-roll inflationary eras; but that typically these suffer from unstable directions in the complete field space.² In spite of this rather negative conclusion, we believe our discussion remains of interest; for example we present a particularly simple form for the scalar potential in a wide class of theories.

²For the original NMSSM-based model of Ref. [24] this was pointed out in Ref. [25].

4 The scalar potential

The scalar part of the $N = 1$ supergravity Lagrangian is given (in the Einstein frame) by

$$\mathcal{L} = g_a{}^b g^{\mu\nu} D_\mu \phi^a D_\nu \phi_b^* - V(\phi, \phi^*) \quad (4.1)$$

where ϕ^a is the chiral scalar multiplet in an arbitrary representation, and $g_a{}^b$ is the Kähler metric, given by

$$g_a{}^b = \frac{\partial^2 K}{\partial \phi^a \partial \phi_b^*}, \quad (4.2)$$

where $K(\phi, \phi^*)$ is the Kähler potential.

The scalar potential V is

$$V = V_F + V_D, \quad (4.3)$$

where

$$V_D = \frac{1}{2} g^2 \text{Re } f_{AB} K_a (R^A)^a{}_b \phi^b K_c (R^B)^c{}_d \phi^d \quad (4.4)$$

and

$$V_F = e^K \left[(g^{-1})_a{}^b (D^a W^*) D_b W - 3 W W^* \right] \quad (4.5)$$

where

$$D_b W \equiv W_b + K_b W, \text{ and } W_b \equiv \frac{\partial W}{\partial \phi^b}, \quad (4.6)$$

and $W(\phi^a)$ is the superpotential. Note that

$$D^a W^* = \frac{\partial W^*}{\partial \phi_a^*} + \frac{\partial K}{\partial \phi_a^*} W^* = (D_a W)^*. \quad (4.7)$$

Consider the Kähler potential defined by

$$K = -3M_P^2 \log[|\Omega(\phi^a, \phi_b^*)|/3], \quad (4.8)$$

with

$$\Omega = \frac{1}{M_P^2} \left[\sum \phi_a^* \phi^a - \frac{1}{2} \xi (c_{ab} \phi^a \phi^b + c.c.) - 3M_P^2 \right]. \quad (4.9)$$

where c_{ab} is an invariant tensor of the gauge group. Of course c_{ab} exists only for certain representations. For example, in $SU(5)$ with two Higgs multiplets, one (H_u) in the **5** and the other (H_d) in the $(\bar{\mathbf{5}})$ or, for the adjoint representation, where $c_{ab} \propto \delta_{ab}$, to which we will presently specialise. Each such independent invariant can be associated with a different non-minimal coupling constant, which we absorb into c_{ab} .

Ω is of course real. Apart from the c_{ab} term, this form of K is precisely that to be found in Wess and Bagger [42], for the potential in the Einstein frame, except that in that reference K is defined as follows:

$$K = -3M_P^2 \log[-\Omega(\phi^a, \phi_b^*)/3]. \quad (4.10)$$

Of course for $c_{ab} = 0$ and $|\phi|^2 < 3M_P^2$, $\Omega < 0$ and Eqs. (4.8), (4.10) are equivalent; we have introduced Eq. (4.8) because we will encounter cases when $\Omega > 0$.

It was shown in Ref. [24] that inclusion of Eq. (3.1) (in the Jordan frame) leads to the form of K (in the Einstein frame) given in Eq. (4.8).

Now the BS paradigm is to have $\xi \gg 1$, so that the ξ term is generally larger than the $\phi^* \phi$ term; it is then important that Ω does not change sign during the inflationary era. In Ref. [24] we presented an example where having $V > 0$ during inflation was incompatible with this requirement.

Reverting again to setting $M_P \equiv 1$, the Kähler metric is

$$g_a{}^b = \frac{\partial^2 K}{\partial \phi^a \partial \phi_b^*} = -\frac{3}{\Omega} \delta_a{}^b + \frac{3}{\Omega^2} \Omega_a \Omega^b = -\frac{3}{\Omega} \delta_a{}^b + \frac{3}{\Omega^2} (\phi_a^* - \xi c_{ac} \phi^c) (\phi^b - \xi c^{bc} \phi_c^*) \quad (4.11)$$

where

$$\Omega_a = \frac{\partial \Omega}{\partial \phi^a}, \quad \Omega^a = \frac{\partial \Omega}{\partial \phi_a^*} \quad (4.12)$$

and $c^{ab} = (c_{ab})^*$. The inverse of the metric is (see Appendix B)

$$\begin{aligned} (g^{-1})_a{}^b &= -\frac{\Omega}{3} \left(\delta_a{}^b - \frac{\Omega_a \Omega^b}{\Omega D} \right) \\ &= -\frac{\Omega}{3} \left(\delta_a{}^b - \frac{(\phi_a^* - \xi c_{ac} \phi^c) (\phi^b - \xi c^{bc} \phi_c^*)}{\Omega D} \right) \end{aligned} \quad (4.13)$$

where

$$\Omega D = \Omega^a \Omega_a - \Omega = \xi^2 c_{ab} \phi^b c^{ac} \phi_c^* - \frac{\xi}{2} (c_{ab} \phi^a \phi^b + c.c.) + 3. \quad (4.14)$$

Substituting our choice of K in Eq. (4.5) we obtain the following surprisingly simple formula:

$$V_F = \frac{9}{\Omega^2} \left[\left| \frac{\partial W}{\partial \phi^a} \right|^2 - \frac{1}{\Omega D} \left| \Omega^a \frac{\partial W}{\partial \phi^a} - 3W \right|^2 \right] \quad (4.15)$$

where $\Omega^a = \phi^a - \xi c^{ab} \phi_b^*$.

Obviously interesting is the special class of trajectories such that

$$\Omega^a \frac{\partial W}{\partial \phi^a} = 3W \quad (4.16)$$

when V_F is positively semi-definite. The minima of the potential on such a trajectory all correspond to zero cosmological constant, $V = 0$, if there exist solutions to the equations

$$\frac{\partial W}{\partial \phi^a} = 0. \quad (4.17)$$

These do not necessarily correspond to local minima of the full potential V_F , however, since the second term in Eq. (4.15) may well be negative in the neighbourhood of a solution to Eq. (4.16). Other than in the global supersymmetry case they also break supersymmetry (unless $W = 0$).

5 Scale invariant superpotentials

In the case when W is a purely cubic superpotential, V_F simplifies even more remarkably to the following form:

$$V_F = \frac{9}{\Omega^2} \left[\left| \frac{\partial W}{\partial \phi^a} \right|^2 - \frac{|\Delta|^2}{\Omega D} \right] \equiv \frac{9}{\Omega^2} (V_1 + V_2), \quad (5.1)$$

where

$$\Delta = \xi \frac{\partial W}{\partial \phi^a} c^{ab} \phi_b^*, \quad \Omega D = \Omega^a \Omega_a - \Omega, \quad (5.2)$$

and we have defined V_1 and V_2 for later convenience.

5.1 The $\xi = 0$ case

The result Eq. (5.1) is interesting even in the case of minimal coupling when $\xi = 0$. Then V_F is manifestly positive, with extrema corresponding to $V_F = 0$ if there exist solutions to

$$\frac{\partial W}{\partial \phi^a} = 0 \quad (5.3)$$

such that $\Omega \neq 0$.

There are three cases to consider.

- Global supersymmetry, i.e. $g_a{}^b = \delta_a{}^b$ in Eq. (4.1), and

$$V_F = \left| \frac{\partial W}{\partial \phi^a} \right|^2. \quad (5.4)$$

In this case a solution to Eq. (5.3) corresponds to a supersymmetric ground state, assuming the D -term also vanishes. (We review in Appendix C the fact that an F -flat potential is also in general D -flat, unless $W = 0$.) Moreover it is easily seen that these correspond to the only possible extrema of V_F ; that is, there are no non-supersymmetric extrema. This is because such extrema would satisfy

$$\frac{\partial^2 W}{\partial \phi^a \partial \phi^b} \frac{\partial W^*}{\partial \phi_b^*} = 0. \quad (5.5)$$

Now this is a set of homogeneous polynomial equations with each and every term of the form $\phi(\phi^*)^2$. Consequently if this system has a solution $\phi = \phi_0$, then it will also have a solution $\phi = \lambda \phi_0$. But if $\phi \rightarrow \lambda \phi$, then $V_F \rightarrow \lambda^4 V_F$. So the only possible extrema have $V_F = 0$.

- Normal supergravity with $\Omega = \phi^* \phi - 3$.

Now an extremum satisfying Eq. (5.3) with $\Omega \neq 0$ again has $V_F = 0$ but now corresponds to *broken* supersymmetry, unless also $W = 0$. In this case (whether or not there are such extrema) there may exist extrema with $V \neq 0$ because the extremal condition is no longer homogeneous. Obviously, without a ξ term, any new extrema will have $\phi \sim M_P$ and so higher order terms in the potentials become significant, so it is not clear what physics we can extract from this case.

- Scale invariant supergravity with $\Omega = \phi^* \phi$.

Once again an extremum satisfying Eq. (5.3) with $\Omega \neq 0$ again has $V_F = 0$ and corresponds to *broken* supersymmetry, unless also $W = 0$. Now, however, there may also be $V_F \neq 0$ extrema, because the potential is invariant under the rescaling $\phi \rightarrow \lambda \phi$. Of course since in this case we have neither an Einstein $M_P^2 R$ term nor a ξ term, we are now describing a theory in a background gravitational field.

As in the non-supersymmetric case, let us consider the case of a single adjoint representation of $SU(N)$. The most general cubic superpotential is

$$W = \frac{\sqrt{2}}{3} \lambda \text{Tr} \Phi^3 = \frac{1}{3} \lambda d^{ABC} \phi^A \phi^B \phi^C \quad (5.6)$$

where we can choose λ to be real and positive, and our normalisation of d^{ABC} is the conventional one for $SU(N)$; see Appendix A. The crucial difference from the non-supersymmetric case is that ϕ^A are complex fields and correspondingly Φ is not hermitian (although still traceless), and therefore cannot be made real and diagonal by a gauge transformation. However, if we assume that the extrema occur for $V_D = 0$, then

$$[\Phi, \Phi^\dagger] = 0. \quad (5.7)$$

Then any Φ satisfying Eq. (5.7) can be diagonalised by a unitary transformation; so we may seek solutions of the form

$$\Phi = \text{diag}(z_1 \cdots z_i \cdots z_N) \quad \text{and traceless}, \quad (5.8)$$

but we cannot in the supersymmetric case assume that z_i are real, without loss of generality.

Returning to the three cases described in section 5.1 we have in turn:

- Global supersymmetry.

Eq. (5.3) gives

$$\Phi^2 - \frac{1}{N} T_2 = 0 \quad (5.9)$$

then we find that z_i satisfy the equation

$$z^2 = \frac{1}{N} T_2 \quad (5.10)$$

where $T_2 = \sum_i z_i^2$. Since this is a quadratic it has at most two solutions, $z_i = \pm \sqrt{\frac{T_2}{N}}$, so we see that if N is odd there is a unique solution $z_i = 0$ for all i , corresponding to a supersymmetric ground state. If N is even, one has an additional supersymmetric ground state

$$\Phi = \Lambda \text{diag}(1, -1, \cdots, 1, -1), \quad (5.11)$$

(where Λ is complex in general), corresponding to the breaking $SU(N) \rightarrow SU(\frac{N}{2}) \otimes SU(\frac{N}{2})$.

- Normal supergravity with $\Omega = \phi^* \phi - 3$.

In this case the scalar potential is

$$V_F = \frac{18\lambda^2}{\Omega^2} \left[\tilde{T}_4 - \frac{1}{N} |T_2|^2 \right] \quad (5.12)$$

where $\tilde{T}_4 = \text{Tr} \Phi^2 \Phi^{\dagger 2}$, $\Omega = \tilde{T}_2 - 3$, with $\tilde{T}_2 = \text{Tr} \Phi \Phi^\dagger$. If we again seek a solution of the form of Eq. (5.8) then we find

$$zz^{*2} - \frac{1}{N} T_2^* z - \frac{1}{\Omega} z^* \left(\tilde{T}_4 - \frac{1}{N} |T_2|^2 \right) - \frac{1}{N} \tilde{T}_3^* = 0 \quad (5.13)$$

where $\tilde{T}_3^* = \text{Tr} \Phi \Phi^{\dagger 2}$. Even for low values of N , it is difficult to analyse the general complex solutions of this equation. For real $z \rightarrow x$, Eq. (5.13) reduces to

$$x^3 - \frac{1}{T_2 - 3} \left(T_4 - 3 \frac{T_2}{N} \right) x - \frac{1}{N} T_3 = 0. \quad (5.14)$$

It is easy to show that for even N , Eq. (5.14) is satisfied by Eq. (5.11). Moreover it follows in general from Eq. (5.14) that $T_4 = T_2^2/N$, so there are no other non-trivial solutions of Eq. (5.14). Therefore, for odd N , there is a unique supersymmetric ground state with $\Phi = V_F = 0$; while for even N there is an additional supersymmetric extremum satisfying Eq. (5.11) and $V_F = 0$, breaking $SU(N) \rightarrow SU(\frac{N}{2}) \otimes SU(\frac{N}{2})$. This extremum is also supersymmetric since $W = 0$.

- Scale invariant supergravity with $\Omega = \phi^* \phi$.

In this case the scalar potential is

$$V_F = \frac{18\lambda^2}{\tilde{T}_2^2} \left[\tilde{T}_4 - \frac{1}{N} |T_2|^2 \right] \quad (5.15)$$

leading to

$$zz^{*2} - \frac{1}{N} T_2^* z - \frac{1}{\tilde{T}_2} z^* \left(\tilde{T}_4 - \frac{1}{N} |T_2|^2 \right) - \frac{1}{N} \tilde{T}_3^* = 0. \quad (5.16)$$

In this case V_F is not well defined at $\Phi = 0$. As usual, it is easier to examine this equation in the special case of real $z \rightarrow x$:

$$x^3 - \frac{T_4}{T_2} x - \frac{1}{N} T_3 = 0. \quad (5.17)$$

Since this is a cubic there are at most three distinct solutions for x_i . For even N we again have the supersymmetric extremum Eq. (5.11), with $V_F = 0$.

A general classification of even the real extrema is more complicated in this case. Let us consider $SU(5)$. Suppose one of the x_i is zero. It follows from Eq. (5.17) that $T_3 = 0$. It is therefore easy to see that the possible solutions for $SU(5)$

are $\Phi = \text{diag}(1, -1, 0, 0, 0)$, $\Phi = \text{diag}(1, -1, 1, -1, 0)$. with results for V of $V_F^{(1)} = 27\lambda^2/5$ and $V_F^{(1)} = 9\lambda^2/10$ respectively.

If none of the x_i are zero then we have the possible forms

$$A : \Phi = \text{diag}(1, 1, 1, z, -3 - z) \quad (5.18)$$

$$B : \Phi = \text{diag}(1, 1, z, z, -2 - 2z). \quad (5.19)$$

Substitution of these forms in Eq. (5.17) and plotting V as a function of z (or seeking consistent solutions for all x_i of Eq. (5.17)) reveals that there are in fact solutions corresponding to

$$A : \Phi = \text{diag}(1, 1, 1, -3/2, -3/2) \quad (5.20)$$

$$B : \Phi = \text{diag}(1, 1, 1, 1, -4) \quad (5.21)$$

with $V_A = 3\lambda^2/5$ and $V_B = 81\lambda^2/10$ respectively.

Solution A from Eq. (5.20), corresponding to breaking to $SU(3) \otimes SU(2) \otimes U(1)$, has the lowest energy. In Appendix D, we show that this is in fact stable against quadratic fluctuations.

5.2 The $\xi \neq 0$ case

The $\xi \neq 0$ case is different because V_F is no longer positive definite. However, we show in Appendix E that the leading term in ξ is nonnegative in scale invariant models.

Let us return to the single adjoint case, Eq. (5.6). It is interesting that the form of the results are quite similar in some cases to the $\xi = 0$ case.

For real fields ϕ , the potential becomes (for $SU(N)$)

$$V_F = \frac{18\lambda^2}{[(1 - \xi)T_2 - 3]^2} \left[T_4 - \frac{1}{N}T_2^2 - \frac{\xi^2 T_3^2}{\xi(\xi - 1)T_2 + 3} \right]. \quad (5.22)$$

In the large ξ limit then, for real ϕ , V_F becomes

$$V_F = \frac{18\lambda^2}{\xi^2 T_2^2} \left[T_4 - \frac{1}{N}T_2^2 - \frac{T_3^2}{T_2} \right]. \quad (5.23)$$

Once again we assume that vanishing of V_D implies Eq. (5.7), and hence a diagonal Φ , so we seek an extremum of V_F with Φ of the form

$$\Phi = \text{diag}(x_1 \cdots x_i \cdots x_N) \quad \text{and traceless.} \quad (5.24)$$

We find that x_i satisfy the cubic equation

$$2x^3 T_2^2 - 3T_3 T_2 x^2 + (3T_3^2 - 2T_4 T_2)x + \frac{1}{N}T_3 T_2^2 = 0. \quad (5.25)$$

Since this is a cubic there are at most three distinct solutions for x_i .

Suppose one of the x_i is zero. It follows from Eq. (5.25) that $T_3 = 0$. It is therefore easy to see that the potential solutions for $SU(5)$ are $\Phi = \text{diag}(1, -1, 0, 0, 0)$, $\Phi = \text{diag}(1, -1, 1, -1, 0)$. with results for V_F of $V_F = 27\lambda^2/(5\xi^2)$ and $V_F = 9\lambda^2/(10\xi^2)$ respectively.

In both the above cases the potential is unstable against quadratic fluctuations. One can see this simply by substituting, for example, $\Phi = \text{diag}(1+Y, -1+Y, -2Y, 0, 0)$ in the first case, or $\Phi = \text{diag}(1, -1, 1, -1-X, X)$ in the second.

If none of the x_i are zero then we have the possible forms

$$A : \Phi = \text{diag}(1, 1, 1, z, -3-z) \quad (5.26)$$

$$B : \Phi = \text{diag}(1, 1, z, z, -2-2z). \quad (5.27)$$

Substitution of these forms in Eq. (5.25) and plotting V as a function of z (or seeking consistent solutions for all x_i of Eq. (5.25)) reveals that there are in fact solutions corresponding to

$$A : \Phi = \text{diag}(1, 1, 1, -3/2, -3/2) \quad (5.28)$$

$$B : \Phi = \text{diag}(1, 1, 1, 1, -4) \quad (5.29)$$

with in both cases $V_F^{(1)} = 0$.

Once again, the $SU(3) \otimes SU(2) \otimes U(1)$ trajectory is stable against quadratic fluctuations. For example, if we set

$$\Phi = \text{diag}(2A + X + Y, 2A - X + Y, 2A + Y, -3A + Y, -3A - 4Y) \quad (5.30)$$

and expand the full potential as a power series in (X, Y) we get

$$\begin{aligned} V_F = & \lambda^2 \frac{18(-450A^4\delta + 4500A^6)\delta^3}{5(30A^2\delta - 30A^2 - 3\delta)^2(30A^2\delta - 30A^2 - 3\delta^2)} \\ & + \lambda^2\delta^2\left(\frac{2}{A^2}X^2 + \frac{25}{2A^2}Y^2\right) + \dots \end{aligned} \quad (5.31)$$

where here $\delta = 1/\xi$, and dropping terms of $O(\delta^3)$ from the X^2, Y^2 terms. For $\xi A^2 \gg 1$ this becomes

$$V_F \sim -\frac{3}{5\xi^3}\lambda^2 + \frac{\lambda^2}{\xi^2}\left(\frac{2}{A^2}X^2 + \frac{25}{2A^2}Y^2\right). \quad (5.32)$$

It appears that the leading $1/\xi^3$ term comes entirely from the $|W|^2$ term in V , as its sign flips if we flip the sign of this term. Obviously, however, making ξ negative corresponds (for large ξA^2) to $\Omega > 0$ and hence we will encounter a zero in Ω as A decreases which is clearly problematic, unless we abandon matching the large ϕ and small ϕ regions.

Notice that the result for $V \sim M_P^4/\xi^3$ is $O(M_{\text{GUT}}^4)$ if $\xi \sim 10^4$. So perhaps we could have a (larger) positive cosmological constant of $O(M_{\text{GUT}}^4)$, which would allow us to have inflation and have that cancelled by the $SU(5)$ breaking? But apart from the fine-tuning issue, if the $SU(5)$ breaking preserves supersymmetry then it would generate a cosmological constant that was naturally of $O(M_{\text{GUT}}^6/M_P^2)$.

6 Scale Invariant Case

It is interesting to consider the scale invariant case, corresponding in the Jordan frame to the absence of the Ricci scalar term, $M_P^2 R$.

We can recover this from the more general results by simply replacing Ω by

$$\Omega = \frac{1}{M_P^2} \left[\sum \phi_a^* \phi^a - \frac{1}{2} \xi (c_{ab} \phi^a \phi^b + c.c.) \right] \quad (6.1)$$

and hence ΩD by

$$\Omega D = \xi^2 c_{ab} \phi^b c^{ac} \phi_c^* - \frac{\xi}{2} (c_{ab} \phi^a \phi^b + c.c.). \quad (6.2)$$

We find from Eq. (4.5) that (for a scale invariant, *i.e.*, cubic, superpotential), the scalar potential still satisfies Eqs. (5.1), (5.2).

6.1 The adjoint case

We return to the single adjoint case, that is the potential in Eq. (5.6). We find from Eq. (5.1)

$$V_F = \frac{18\lambda^2}{\Omega^2} \left[\tilde{T}_4 - \frac{1}{N} |T_2|^2 - \frac{\xi |\tilde{T}_3|^2}{\xi \tilde{T}_2 - \Re(T_2)} \right] \quad (6.3)$$

where $\tilde{T}_4 \equiv \text{Tr}[\Phi^{\dagger 2} \Phi^2]$, $\tilde{T}_3 \equiv \text{Tr}[\Phi^\dagger \Phi^2]$. It is easy to check that in the large ξ limit Eq. (6.3) reduces to Eq. (5.23); the large ξ limit is the same for the scale invariant case.

If we again specialise to the case of a real-valued extrema of V_F with Φ of the form of Eq. (5.8), then, so long as $T_2 \neq 0$, we find that x_i satisfy the cubic equation

$$2x^3 T_2^2 - 3\rho T_3 T_2 x^2 + (3\rho T_3^2 - 2T_4 T_2)x + \frac{3\rho - 2}{N} T_3 T_2^2 = 0 \quad (6.4)$$

where we defined $\rho \equiv \xi/(\xi - 1) > 1$. It is also interesting to ask what the condition for a supersymmetric state is. Again assuming Eq. (5.8), we find that in supersymmetric states the x_i satisfy the quadratic equation

$$x^2 - \frac{T_3}{T_2} x - \frac{1}{N} T_2 = 0 \quad (6.5)$$

with non-vanishing diagonal $N \times N$ solutions of the form

$$\begin{aligned} \Phi \sim & (1, -1), \\ & (1, 1, -2), \\ & (1, 1, -1, -1), (1, 1, 1, -3), \\ & (2, 2, 2, -3, -3), (1, 1, 1, 1, -4) \dots \end{aligned} \quad (6.6)$$

(This is reminiscent of the flat space case for a massive adjoint:

$$W = \frac{1}{2} m T_2 + \frac{\lambda}{3} T_3 \quad (6.7)$$

when the x_i also satisfy a quadratic

$$\lambda x^2 + mx - \frac{\lambda}{N} T_2 = 0, \quad (6.8)$$

but of course in that case the solutions are not scale invariant, and $\Phi = 0$ is also a solution.)

In the $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$ case, it is easy to show from Eq. (6.3) that

$$V_F = -\frac{3M_P^4 \lambda^2}{5(\xi - 1)^3}. \quad (6.9)$$

Turning to the case of more general (i.e. not necessarily supersymmetric) extrema, we require a solution to Eq. (6.4). If $T_3 = 0$, then this reduces to

$$x(x^2 T_2 - T_4) = 0 \quad (6.10)$$

with solutions corresponding to

$$\begin{aligned} \Phi \sim & (1, -1, 0), (1, -1, 0, 0), (1, -1, 0, 0, 0) \dots \\ & (1, -1, 1, -1, 0), (1, -1, 1, -1, 0, 0), (1, -1, 1, -1, 0, 0, 0) \dots \\ & (1, -1, 1, -1, 1, -1, 0), (1, -1, 1, -1, 1, -1, 0, 0), \dots \end{aligned} \quad (6.11)$$

etc, as well as the supersymmetric solutions $\Phi \sim (1, -1)$ etc. identified above. The extrema identified in Eq. (6.11) all correspond to $W = 0$ and give a positive V_F and supersymmetry breaking.

For $SU(N)$, one can characterise a diagonal form of Φ giving a potential extremum in terms of 3 parameters m, n, z as follows:

$$\Phi = \text{diag} \left(\underbrace{1, \dots, 1}_m, \underbrace{z, \dots, z}_n, \underbrace{-\frac{m+nz}{N-m-n}, \dots, -\frac{m+nz}{N-m-n}}_{N-m-n} \right), \quad (6.12)$$

where there are m entries of 1 and n entries of z .

6.1.1 The adjoint case for $SU(5)$

For $N = 5$ it is straightforward to list and investigate the various possible results for Φ of the form of Eq. (6.12) which correspond to extrema of the potential

- For the case $N = 5$, $m = 3$, $n = 1$ (or, equivalently, $m = 3$, $n = 2$), we find that, as well as supersymmetric solutions for $z = 1$ and $z = -3/2$, there are supersymmetry breaking solutions for

$$z = z_{A,A'} = -\frac{3}{2} \pm \frac{1}{2} \sqrt{\frac{15(8-3\rho)}{27\rho-8}}, \quad (6.13)$$

and we assume that $1 < \rho < 8/3$.

These extrema are not local minima for every direction in field space. For example if we set

$$\Phi = \text{diag}(1, 1, 1, z_A + X, -3 - z_A - X) \quad (6.14)$$

Then (for large ξ)

$$V_F = \frac{25}{18\xi^2} - \frac{6859}{2592\xi^2}X^2 + \dots, \quad (6.15)$$

and so these extrema are unstable. It is interesting that the solutions Eq. (6.13) yields $V_F = 0$ for the specific values $\xi = (152 \pm 72\sqrt{6})/125 \approx \pm 2.63$. However this supersymmetry-breaking solution is unstable in the same direction as described above in Eq. (6.14).

- For the case $N = 5$, $m = 2$, $n = 1$ (or, equivalently, $m = 2$, $n = 2$), we find that, in addition to supersymmetric solutions for $z = 1$ and $z = -2/3$, there are supersymmetry breaking solutions as follows:

(a) $z = 0$. This is unstable with respect to fluctuations for example if

$$\Phi = \text{diag}(1, 1, X, -1 - X, -1) \quad (6.16)$$

We find (for large ξ)

$$V_F = \frac{9}{10\xi^2} - \frac{45}{32\xi^2}X^2 + \dots \quad (6.17)$$

(b)

$$z = z_{B,B'} = \frac{9\rho + 4 \pm \sqrt{5(9\rho + 4)(9\rho - 4)}}{3(3\rho - 2)}, \quad (6.18)$$

where again we take $\rho > 1$. These are unstable with respect to

$$\Phi = \text{diag}(1, 1 - X, z_B + X, -1 - \frac{z_B}{2}, -1 - \frac{z_B}{2}) \quad (6.19)$$

We find (for large ξ)

$$V_F \approx \frac{25}{9\xi^2} - \frac{0.15}{\xi^2}X^2 + \dots \quad (6.20)$$

- The only possible case not included in the 2 cases above (when permutations are included) is $m = 1$, $n = 1$, leading to $z = -1$. (or, equivalently, $m = 1$, $n = 3$, $z = 0$.) This is unstable with

$$\Phi = \text{diag}(1 + X, -1, -X, 0, 0). \quad (6.21)$$

We find (for large ξ)

$$V_F = \frac{27}{5\xi^2} - \frac{81}{4\xi^2}X^2 + \dots \quad (6.22)$$

For general ξ the corresponding expression is

$$V_F = \frac{27}{5(\xi - 1)^2} - \frac{81\xi}{4(\xi - 1)^3}X^2 + \dots \quad (6.23)$$

so that the instability in fact persists for all $\xi > 1$. This is in fact true for all the cases described in this subsection.

7 The Witten Model

In this section we consider a variation of the inverted hierarchy model of Witten [43]-[46], defined by the superpotential

$$\begin{aligned}
W &= \frac{\lambda_1}{2} d^{ABC} A^A A^B Y^C + \frac{\lambda_2}{2} X (A^A A^A - m^2) \\
&= \frac{\lambda_1}{\sqrt{2}} \text{Tr}(A^2 Y) + \frac{\lambda_2}{2} X (\text{Tr} A^2 - m^2) \\
&= \bar{\lambda}_1 \text{Tr}(A^2 Y) + \bar{\lambda}_2 X (\text{Tr} A^2 - m^2)
\end{aligned} \tag{7.1}$$

where A, Y are $SU(5)$ adjoints and X is a singlet, and $\bar{\lambda}_{1,2}$ are the couplings as originally defined by Witten. In its complete form, with $m \neq 0$, supersymmetry is broken spontaneously in the O’Raifeartaigh manner; moreover $SU(5)$ is broken to $SU(3) \otimes SU(2) \otimes U(1)$, with the scale at which this occurs being larger than and not directly related to m^2 , generated in fact by dimensional transmutation [44].

At the minimum of the potential it is straightforward to show that

$$A = m \frac{\lambda_2}{\sqrt{\lambda_1^2 + 15\lambda_2^2}} \text{diag}(2, 2, 2, -3, -3) \tag{7.2}$$

and

$$Y = \bar{Y} \text{diag}(2, 2, 2, -3, -3) = \frac{\bar{\lambda}_2}{\lambda_1} X \text{diag}(2, 2, 2, -3, -3) \tag{7.3}$$

with X undetermined in the tree approximation.

Our variation will be to have $m^2 = 0$ in Eq. (7.1), but with the $SU(5)$ breaking still generated in similar fashion³ but we can imagine the supersymmetry breaking provided instead by anomaly mediation, gravity-mediated soft breaking, or even simply a cosmological fluctuation that gives rise to $A \neq 0$.

For $m^2 = 0$, we find in the notation of Eq. (5.1) that

$$\begin{aligned}
V_1 &= \frac{\lambda_1^2}{2} \left[\text{Tr}(\{A^\dagger, Y^\dagger\}\{A, Y\}) - \frac{4}{5} |\text{Tr} A Y|^2 + \text{Tr}(A^{\dagger 2} A^2) - \frac{1}{5} |\text{Tr} A^2|^2 \right] \\
&\quad + \lambda_2^2 \left[|X|^2 \text{Tr} A^\dagger A + \frac{1}{4} |\text{Tr} A^2|^2 \right] + \sqrt{2} \lambda_1 \lambda_2 \Re(X^\dagger \text{Tr} A^\dagger \{A, Y\})
\end{aligned} \tag{7.4}$$

and

$$\Omega = \sum \Phi^* \Phi - \frac{\xi}{2} [\text{Tr} A^2 + c_Y \text{Tr} Y^2 + c_X X^2 + c.c.]. \tag{7.5}$$

We can without loss of generality assume that ξ , c_X and c_Y are real and positive. In fact, we will require that $\Omega < 0$ so that the Kahler potential $K = -3 \log(|\Omega|/3)$ is not singular. This assumption implies, in particular, that $\xi, c_X \xi, c_Y \xi > 1$.

³A discussion of the $m^2 \rightarrow 0$ limit of Witten’s model appears in Ref. [44].

For simplicity, we will assume consider only real values of the fields. Let us first analyse the potential in the approximation that $A \gg X, Y$. Then there is no contribution from V_2 , and

$$V_F = \frac{9}{\Omega^2} V_1 = \frac{9}{(\xi - 1)^2} \left[\frac{\lambda_1^2}{2} \left(\frac{T_4}{T_2^2} - \frac{1}{5} \right) + \frac{\lambda_2^2}{4} \right]. \quad (7.6)$$

If we seek an extremum of V_F with A of the form

$$A = \text{diag}(x_1 \cdots x_i \cdots x_5) \quad \text{and traceless}, \quad (7.7)$$

then we find that x_i satisfy the cubic equation

$$x^3 T_2 - T_4 x - \frac{1}{5} T_3 T_2 = 0. \quad (7.8)$$

Since this is a cubic there are at most three distinct solutions for x_i . Analysing as before we find that the extrema are

$$(1, -1, 0, 0, 0), (1, -1, 1, -1, 0), (2, 2, 2, -3, -3), (1, 1, 1, 1, -4). \quad (7.9)$$

The one with the lowest energy is $(2, 2, 2, -3, -3)$, with

$$V_F = \frac{9}{(\xi - 1)^2} \left(\frac{\lambda_1^2}{60} + \frac{\lambda_2^2}{4} \right), \quad (7.10)$$

and it appears to be stable against fluctuations, in A at least.

Encouraged by this fact, we now turn to the more relevant case that A, X, Y are all large and X, Y satisfy Eq. (7.3). Moreover we will assume that A, Y are parallel, that is that $A = \bar{A} \text{diag}(2, 2, 2, -3, -3)$. Then we find that (notation from Eq. (5.1))

$$V_1 = \lambda_1^2 (60 \bar{A}^2 \bar{Y}^2 + 15 \bar{A}^4) + \lambda_2^2 (30 \bar{A}^2 X^2 + 225 \bar{A}^4) - 60 \sqrt{2} \lambda_1 \lambda_2 X \bar{Y} \bar{A}^2, \quad (7.11)$$

and substituting from Eq. (7.3), the $\bar{A}^2 X^2$ terms all cancel and we get simply

$$V_1 = 15(\lambda_1^2 + 15\lambda_2^2) \bar{A}^4. \quad (7.12)$$

Now we analyse V_2 in similar fashion. From Eq. (5.2) we have that

$$\Delta = 15\xi \left[2(\lambda_2 X - \sqrt{2} \lambda_1 \bar{Y}) \bar{A}^2 + (c_X \lambda_2 X - \sqrt{2} c_Y \lambda_1 \bar{Y}) \bar{A}^2 \right], \quad (7.13)$$

and using Eq. (7.3) again we get

$$\Delta = 15\xi (c_X - c_Y) \lambda_2 X \bar{A}^2. \quad (7.14)$$

Meanwhile,

$$\begin{aligned}\Omega &= -30(\xi - 1)\bar{A}^2 - \left[15(c_Y\xi - 1)\frac{\lambda_2^2}{\lambda_1^2} + (c_X\xi - 1) \right] X^2 \\ &\equiv -30\xi(\rho\bar{A}^2 + \rho'_X X^2),\end{aligned}\tag{7.15}$$

$$\begin{aligned}\Omega D &= 30\xi(\xi - 1)\bar{A}^2 + 30c_Y\xi(c_Y\xi - 1)\bar{Y}^2 + c_X\xi(c_X\xi - 1)X^2 \\ &= 30\xi(\xi - 1)\bar{A}^2 + 30\left[c_Y\xi(c_Y\xi - 1)\frac{\lambda_2^2}{2\lambda_1^2} + \frac{1}{30}c_X\xi(c_X\xi - 1) \right] \\ &\equiv 30\xi^2(\rho\bar{A}^2 + \rho_X X^2).\end{aligned}\tag{7.16}$$

For reference, we make the implicit definitions in Eqs. (7.15),(7.16) explicit:

$$\begin{aligned}\rho &\equiv 1 - \frac{1}{\xi}, & \rho'_X &\equiv \left(c_Y - \frac{1}{\xi} \right) \frac{\lambda_2^2}{2\lambda_1^2} + \frac{1}{30} \left(c_X - \frac{1}{\xi} \right), \\ \rho_X &\equiv c_Y \left(c_Y - \frac{1}{\xi} \right) \frac{\lambda_2^2}{2\lambda_1^2} + \frac{1}{30} c_X \left(c_X - \frac{1}{\xi} \right).\end{aligned}\tag{7.17}$$

Noting that

$$\rho_X = \left(c_Y - \frac{1}{\xi} \right)^2 \frac{\lambda_2^2}{2\lambda_1^2} + \frac{1}{30} \left(c_X - \frac{1}{\xi} \right)^2 + \frac{\rho'_X}{\xi},\tag{7.18}$$

it follows that $\rho'_X > 0$ implies $\rho_X > 0$.

Finally, the contribution of the second term in the potential is

$$V_2 = -15 \frac{(c_X - c_Y)^2 \lambda_2^2 X^2 \bar{A}^4}{2(\rho\bar{A}^2 + \rho_X X^2)}.\tag{7.19}$$

Combining this with the result above for V_1 ,

$$V_1 + V_2 = 15\bar{A}^4 \left[\lambda_1^2 + \lambda_2^2 \left(15 - \frac{(c_X - c_Y)^2 X^2}{2(\rho\bar{A}^2 + \rho_X X^2)} \right) \right]\tag{7.20}$$

$$= \frac{15\bar{A}^4}{(\rho\bar{A}^2 + \rho_X X^2)} \left[(\lambda_1^2 + 15\lambda_2^2) \rho\bar{A}^2 + (c_X\lambda_1^2 + 15c_Y\lambda_2^2) \rho'_X X^2 \right].\tag{7.21}$$

The second expression above shows that this is nonnegative for all field values since, as noted earlier, we have $\rho, \rho'_X, \rho_X > 0$. This is a rather surprising result.

We are now ready to analyse the complete potential,

$$V_F = \frac{9}{\Omega^2} (V_1 + V_2)\tag{7.22}$$

$$= \frac{3\bar{A}^4 \left[(\lambda_1^2 + 15\lambda_2^2) \rho\bar{A}^2 + (c_X\lambda_1^2 + 15c_Y\lambda_2^2) \rho'_X X^2 \right]}{20\xi^2(\rho\bar{A}^2 + \rho'_X X^2)^2(\rho\bar{A}^2 + \rho_X X^2)}\tag{7.23}$$

$$= \frac{3\mathcal{A}^4 \left[(\lambda_1^2 + 15\lambda_2^2) \rho\mathcal{A}^2 + (c_X\lambda_1^2 + 15c_Y\lambda_2^2) \rho'_X \right]}{20\xi^2(\rho\mathcal{A}^2 + \rho'_X)^2(\rho\mathcal{A}^2 + \rho_X)}\tag{7.24}$$

In the last step, we took advantage of the scale invariance to write the result in terms of the ratio of field values $\mathcal{A} = \bar{A}/X$.

Although we have $V_F > 0$ in general, it is certainly not flat, *i.e.*, not independent of \mathcal{A} . Note that if $X \rightarrow \infty$ for fixed \bar{A} , or $\mathcal{A} \rightarrow 0$, then $V_F \rightarrow O(\mathcal{A}^4)$. If we return for a moment to the original model, Eq. (7.1), we recall that, as $m^2 \rightarrow 0$, $\bar{A} \rightarrow 0$, but X and \bar{Y} , remain at their values determined by dimensional transmutation. Although the model is supersymmetric in the limit, as remarked earlier, one may simply suppose that, initially $\bar{A} \neq 0$ for whatever reason, and work out the consequences as it relaxes toward its equilibrium value, which has been assumed to be negligible compared to the GUT scale. Thus, it is natural to inquire further into the behaviour of V_F for $\bar{A}/X \ll 1$. We find

$$\left[\frac{135\lambda_1^4(c_X\lambda_1^2 + 15c_Y\lambda_2^2)^2}{(c_X^2\lambda_1^2 + 15c_Y^2\lambda_2^2)(c_X\lambda_1^2 + 15c_Y\lambda_2^4)} \right] \left(\frac{M_P\bar{A}}{\sqrt{\xi}X} \right)^4 + O\left(\left(\frac{\bar{A}}{X} \right)^6 \right), \quad (7.25)$$

up to small corrections of order $1/\xi$. We have restored the unit of mass, M_P that was introduced in performing the conformal transformation to Einstein frame. Since it serves as the effective Planck mass for the gravitational interaction, it takes the value of order $\sqrt{\xi}X$, at least naively, so that V_F is indeed correctly represented in units of the gravitational constant. It is interesting that the coefficient in square brackets in Eq. (7.25) is naturally nonnegative for all values of the couplings, which was not guaranteed *a priori*, so far as we are aware⁴. This result is therefore like a small field inflationary model of the form $\lambda\bar{A}^4$. For such models to have any chance at describing a phenomenologically acceptable inflationary epoch, the coefficient λ must be exceedingly small, which could be arranged but does not seem to be required in the present context. Even if finely tuned, such models have been rather thoroughly investigated⁵ and are essentially ruled by the most recent WMAP data.

At the other extreme, for $\bar{A} \gg X$, or $\mathcal{A} \rightarrow \infty$, we obtain

$$V_F = \frac{3}{20\xi^2\rho^2} \left[a_1 - (a_2 + 2a_1\rho'_X) \frac{1}{\rho\mathcal{A}^2} \right] \quad (7.26)$$

where $a_1 = \lambda_1^2 + 15\lambda_2^2$ and $a_2 = (c_X - c_Y)^2\lambda_2^2/2$. So the large \bar{A} -flat trajectory is unstable against fluctuations in X . On the other hand, although $V_F(\mathcal{A})$ approaches its asymptotic value from below, depending on the values of the various parameters, it is not obviously monotonically increasing. This raises the possibility of a local minimum at some value of the ratio \mathcal{A}_0 , which may provide a flat direction of the form $C(\bar{A} - \mathcal{A}_0X)^2/X^2$, where the overall scale X is undetermined classically but will be determined by dimensional transmutation in higher order. This then might provide a model for large field inflation⁶.

⁴We have not investigated whether the curvature remains positive for complex \bar{A} .

⁵For some discussion and references, see, *e.g.*, Ref. [29].

⁶Of course, if such a minimum exists, it is metastable and will eventually tunnel to smaller values of the fields. So it would be necessary that its lifetime is long enough to allow sufficient expansion, typically on the order of 60 e-folds.

To explore this possibility, we need to calculate the derivatives of V_F and determine the range of parameters that generate such a scenario. We first seek values of \mathcal{A} where V_F is stationary. Since $V_F > 0$ everywhere (except at the origin,) we may equally well consider the variation of $\log(V_F)$. It is convenient to absorb the factor of ρ into the ratio \mathcal{A} and to take advantage of the fact the V_F depends on \mathcal{A} only through even powers. Defining $w \equiv \rho \mathcal{A}^2$ and $a_3 \equiv (c_X \lambda_1^2 + 15 c_Y \lambda_2^2) \rho'_X > 0$, we may express the potential and its first variation as

$$V_F = \frac{3}{20 \xi^2 \rho^2} \left[\frac{w^2 (a_1 w + a_3)}{(w + \rho'_X)^2 (w + \rho_X)} \right] \quad (7.27)$$

$$\frac{\partial \log[V_F]}{\partial w} = \frac{2}{w} + \frac{a_1}{(a_1 w + a_3)} - \frac{2}{(w + \rho'_X)} - \frac{1}{(w + \rho_X)}. \quad (7.28)$$

Thus, we need to solve $\partial \log[V_F]/\partial w = 0$. At first, one might think that the vanishing of the first derivative for $w \neq 0$, requires the solution of a cubic polynomial; however, because V_F asymptotes to a constant, the cubic term is absent, and we only need solve a quadratic equation. We find that the first derivative vanishes for

$$(a_1 (2\rho'_X + \rho_X) - a_3) w^2 + \rho'_X (3a_1 \rho_X + a_3) w + 2a_3 \rho_X \rho'_X = 0. \quad (7.29)$$

There exist real roots if the discriminant $\tilde{\Delta}$ is non-negative:

$$\tilde{\Delta} \equiv \rho'_X (a_3 - a_1 \rho_X) (a_3 (\rho'_X + 8\rho_X) - 9a_1 \rho_X \rho'_X) \geq 0, \quad (7.30)$$

which can be arranged. Then the roots take the values

$$w_{\pm} \equiv \frac{-(a_3 + 3a_1 \rho_X) \rho'_X \pm \sqrt{\tilde{\Delta}}}{2(a_1 (2\rho'_X + \rho_X) - a_3)} \quad (7.31)$$

Naturally, we expect one root to be a maximum and the other to be a minimum of V_F . However, recalling that $w \equiv \rho \mathcal{A}^2$, the minimum must occur at $w > 0$ to be acceptable. Unfortunately, it turns out that both roots are negative. To see this, one may first show, under the assumptions stated previously, that the denominator $(a_1 (2\rho'_X + \rho_X) - a_3)$ is positive, so that obviously $w_- < 0$. In order for $w_+ > 0$, we would have to have $\sqrt{\tilde{\Delta}} > (a_3 + 3a_1 \rho_X) \rho'_X$, or $\tilde{\Delta} - (a_3 + 3a_1 \rho_X)^2 \rho'^2_X > 0$. However, this difference turns out to be equal to $8a_3 \rho_X \rho'_X (a_3 - a_1 (2\rho'_X + \rho_X)) < 0$. Thus, both roots are at negative w , so that in fact, V_F is monotonically increasing throughout the region $w > 0$.

One could imagine variants on this theme involving several of the fields in this model, but, since this sort of model appears to be quite different from the Higgs inflation models we were seeking in this paper, we leave such speculations for future work.

8 Conclusions

We have shown that in a GUT with non-minimal scalar coupling to gravity, an era of Higgs inflation is possible with the relevant Higgs multiplet being the one responsible

for breaking of the GUT symmetry. In particular, in a non-supersymmetric $SU(5)$ with an adjoint multiplet, the flat direction with the lowest energy corresponds to $SU(3) \otimes SU(2) \otimes U(1)$, and is stable against fluctuations away from this direction.

Non-supersymmetric $SU(5)$ has problems; with gauge unification and with proton decay for example. While it might be possible to construct a viable model along these lines, we have in this paper also investigated whether the above result can be achieved in a supersymmetric GUT. There has been considerable work on such theories; in particular on $SO(10)$; there in our opinion, however, no really compelling supersymmetric GUT exists as yet. In this paper we have restricted our attention largely to $SU(5)$; and in all the cases we have looked at, although there have existed positive energy flat directions at large field magnitude, they have invariably been unstable against quadratic fluctuations, in the same manner [25] as the original NMSSM model described in Ref. [24]. This problem has been approached in the literature by including higher order terms in the Kähler potential, an approach that we find unattractive and have tried to avoid. As we indicated above, the result of this “purist” philosophy is, unfortunately, that we have been unable to find a supersymmetric model that has both $V_F > 0$ in a flat direction and is stable to fluctuations in other directions. In our opinion this outcome is likely to persist in generalisations to other gauge symmetry groups.

It seems to us, therefore, a minimal extension of the BS scenario to a GUT is more promising in the non-supersymmetric case. Whether this is worth pursuing further may well depend upon whether the current absence of evidence for low energy supersymmetry at LHC experiments persists.

A Group Theory

We consider a complex scalar multiplet ϕ^a transforming according to a (in general reducible) complex representation of a unitary group (we will presently specialise to $SU(N)$) as follows:

$$\phi^a \rightarrow \phi^{a'} = U^a_b \phi^b \quad (\text{A.1})$$

The complex conjugate of ϕ , $\phi_a^* = (\phi^a)^*$ transforms as

$$\phi_a^* \rightarrow \phi_a^{*'} = (U^{-1})^b_a \phi_b^* \quad (\text{A.2})$$

and of course $\phi_a^* \phi^a$ is invariant. We see that it is helpful to use a notation where complex conjugation raises and lowers the index; a familiar notation for the fundamental representation of $SU(N)$.

For the generators of the group in the ϕ^a representation we use the notation $(R^A)^a_b$, thus

$$U = e^{i\alpha^A R^A} \quad (\text{A.3})$$

and

$$[R^A, R^B] = i f^{ABC} R^C. \quad (\text{A.4})$$

When ϕ^a is in the adjoint representation then obviously $\phi^a \rightarrow \phi^A$, and $(R^A)^b_c \rightarrow -if^{ABC}$.

For a single adjoint representation of $SU(N)$ we would have the invariant tensor $c_{AB} = \delta_{AB}$, so in that case it is tempting to define $\phi_A = \delta_{AB}\phi^B$, but since we associate raising and lowering indices with complex conjugation we will not do this. Thus whereas when we write formulae valid for an arbitrary representation, index summations will always involve one up and one down index (simply because the product of a representation with its complex conjugate always contains a singlet) in the case of the adjoint we also have the invariants $\phi^A\phi^A$ and $\phi_A^*\phi_A^*$.

It is convenient to write an adjoint representation as a N -dimensional matrix, $\Phi = \frac{1}{\sqrt{2}}\lambda^A\phi^A$. where λ^A are the generators in the fundamental representation, but conventionally defined as $\lambda^A = 2R^A$. Thus

$$[\lambda^A, \lambda^B] = 2if^{ABC}\lambda^C. \quad (\text{A.5})$$

The following relations are valid for $SU(N)$:

$$\text{Tr}\Phi^2 = \phi^A\phi^A \quad (\text{A.6})$$

$$\text{Tr}\Phi^3 = \frac{1}{\sqrt{2}}d^{ABC}\phi^A\phi^B\phi^C \quad (\text{A.7})$$

$$\text{Tr}\Phi^4 = \frac{1}{2}d^{ACD}\phi^C\phi^Dd^{ADE}\phi^D\phi^E + \frac{1}{N}(\phi^A\phi^A)^2. \quad (\text{A.8})$$

When we generalise to complex ϕ we will also need

$$\text{Tr}\Phi\Phi^\dagger = \phi^A\phi_A^* \quad (\text{A.9})$$

$$\text{Tr}\Phi^2\Phi^\dagger = \frac{1}{\sqrt{2}}d^{ABC}\phi^A\phi^B\phi_C^* \quad (\text{A.10})$$

$$\text{Tr}\Phi^2\Phi^{\dagger 2} = \frac{1}{2}d^{ACD}\phi^C\phi^Dd^{ADE}\phi_D^*\phi_E^* + \frac{1}{N}|\phi^A\phi^A|^2 \quad (\text{A.11})$$

All these expressions follow easily from the formula

$$\{\lambda^A, \lambda^B\} = 2d^{ABC}\lambda^C + \frac{4}{N}\delta^{AB}. \quad (\text{A.12})$$

B Inverse Metrics

The fact that the inverse of a Kähler metric of the form

$$g_a{}^b = \delta_a{}^b - \frac{1}{\Omega}\Omega_a\Omega^b \quad (\text{B.1})$$

is

$$(g^{-1})_a{}^b = \delta_a{}^b - \frac{\Omega_a\Omega^b}{\Omega D} \quad (\text{B.2})$$

where

$$D = \sum \frac{|\Omega_a|^2}{\Omega} - 1 \quad (\text{B.3})$$

is easily derived using the fact that

$$\Pi_1 \equiv \delta_a^b - \frac{\Omega_a \Omega^b}{\sum |\Omega_c|^2} \quad (\text{B.4})$$

and

$$\Pi_2 \equiv \frac{\Omega_a \Omega^b}{\sum |\Omega_c|^2} \quad (\text{B.5})$$

are projection operators,

$$(\Pi_1)^2 = \Pi_1, (\Pi_2)^2 = \Pi_2, \Pi_1 \Pi_2 = 0, \Pi_1 + \Pi_2 = 1. \quad (\text{B.6})$$

Thus

$$\begin{aligned} g_a^b &= \delta_a^b - \frac{\Omega_a \Omega^b}{\sum |\Omega_c|^2} + \frac{\Omega_a \Omega^b}{\sum |\Omega_c|^2} - \frac{\Omega_a \Omega^b}{\Omega} \\ &= \Pi_1 + \left(\frac{\Omega - \sum |\Omega_c|^2}{\Omega} \right) \Pi_2 \end{aligned} \quad (\text{B.7})$$

whence Eq. (B.2) follows, using

$$(a_1 \Pi_1 + a_2 \Pi_2)^{-1} = \frac{1}{a_1} \Pi_1 + \frac{1}{a_2} \Pi_2. \quad (\text{B.8})$$

It is easy to verify that

$$g_a^b (g^{-1})_b^c = (g^{-1})_a^b g_b^c = \delta_a^c. \quad (\text{B.9})$$

It is interesting (if not immediately relevant to our considerations here) to generalise the above case to the problem of finding the inverse of a matrix whose components are either the identity δ^{ab} or the outer product of the vector $\{A^a, A^{*a}\}$ with itself.

For this discussion raising and lowering indices by complex conjugation is no longer convenient. Consider the matrix

$$g^{ab} = \sum_i a_i P_i \quad (\text{B.10})$$

where

$$P_1 = \delta^{ab}, P_2 = A^a A^{*b}, P_3 = A^{*a} A^b, P_4 = A^a A^b, P_5 = A^{*a} A^{*b}. \quad (\text{B.11})$$

The coefficients a_i may depend on scalar invariants involving this vector, such as Ω . The matrix g^{ab} is hermitian if $a_{1,2,3}$ are real and $a_5 = a_4^*$.

It is easy to construct a multiplication table for the P_i : In Table 1, $\alpha = A^{*a} A^a$, $\beta = A^{*a} A^{*a}$, $\gamma = A^a A^a$, and the (i, j) element of the array is $P_i P_j$.

Armed with this table it is straightforward to construct the inverse of g^{ab} , it is

$$(g^{-1})^{ab} = \sum_i b_i P_i \quad (\text{B.12})$$

	P_1	P_2	P_3	P_4	P_5
P_1	P_1	P_2	P_3	P_4	P_5
P_2	P_2	αP_2	βP_4	αP_4	βP_2
P_3	P_3	γP_5	αP_3	γP_3	αP_5
P_4	P_4	γP_2	αP_4	γP_4	αP_2
P_5	P_5	αP_5	βP_3	αP_3	βP_5

Table 1. Multiplication Table

where

$$\begin{aligned}
b_1 &= \frac{1}{a_1} \\
b_2 &= -\frac{a_1 a_2 + \alpha a_2 a_3 - \alpha a_4 a_5}{\Delta} \\
b_3 &= -\frac{a_1 a_3 + \alpha a_2 a_3 - \alpha a_4 a_5}{\Delta} \\
b_4 &= -\frac{a_1 a_4 - \beta a_2 a_3 + \beta a_4 a_5}{\Delta} \\
b_5 &= -\frac{a_1 a_5 - \gamma a_2 a_3 - \gamma a_4 a_5}{\Delta}
\end{aligned} \tag{B.13}$$

and

$$\Delta = a_1[a_1^2 + \alpha a_1 a_2 + \gamma a_1 a_4 - \beta \gamma a_2 a_3 + \beta a_1 a_5 + \beta \gamma a_4 a_5 + \alpha^2 a_2 a_3 + \alpha a_1 a_3 - \alpha^2 a_4 a_5] \tag{B.14}$$

The inverse Eq. (B.2) of the metric Eq. (B.1) is easily derived, by setting $a_1 = 1$, $a_2 = -1/\Omega$, $a_{3,4,5} = 0$.

C F and D -flatness

The conditions for an unbroken supersymmetric state are

$$F_i = D^a = 0 \tag{C.1}$$

where on a curved background,

$$F_i = W_i + K_i W, \tag{C.2}$$

and

$$D^a = G_i (R^a)^i_j \phi^j \tag{C.3}$$

where

$$G = K + \ln W + \ln W^* \tag{C.4}$$

We see that

$$G_i = (1/W) F_i \tag{C.5}$$

so that unless $W = 0$, F -flatness implies D -flatness.

On flat space the argument is more tricky. Here we have

$$F_i = W_i \quad (\text{C.6})$$

and

$$D^a = \phi_i^* (R^a)^i_j \phi^j \quad (\text{C.7})$$

and vanishing of F_i does not seem to tell us much about D^a .

However, consider the gauge transformation $\phi \rightarrow \phi' = U\phi$. We have $W(\phi') = W(\phi)$ and hence $(F_i)' = 0$ if $F_i = 0$. But since W is holomorphic we can transform $\phi^* \rightarrow \phi^{*'} = V\phi^*$ for $V \neq U$.

Consequently

$$D^a \rightarrow \phi^\dagger V^\dagger (R^a) U \phi \quad (\text{C.8})$$

and U, V can be chosen so that D^a transforms to zero. This is easy to see; given a gauge invariant polynomial $P(\phi)$ it is trivial that

$$\frac{\partial P}{\partial \phi} R \phi = 0, \quad (\text{C.9})$$

and so we simply have to choose $\frac{\partial P}{\partial \phi}$ so that

$$\frac{\partial P}{\partial \phi} = \phi^\dagger V^\dagger \quad (\text{C.10})$$

D Stability of the Scale-Invariant, $SU(5)$ Minimum

In this appendix, we show that the lowest energy extremum, Solution A from Eq. (5.20), is in fact a local minimum. In order to enforce the trace constraint, $\sum z_j = 0$, the minimum of the scalar potential in Eq. (5.15) may be obtained by the method of Lagrange multipliers. We seek to find extrema of the auxiliary function G ,

$$G(z, z^*) = V_F(z, z^*) - C^* \sum z_j - C \sum z_j^*. \quad (\text{D.1})$$

As usual, when minimising G , we ignore the constraint condition on the variables, and then choose the (complex) constant C so as to enforce the trace constraint. Thus,

$$\frac{\partial G}{\partial z_k} = \frac{\partial V_F}{\partial z_k} - C^* = 0, \quad (\text{D.2})$$

$$\frac{\partial V_F}{\partial z_k} = \frac{18\lambda^2}{\tilde{T}_2^2} \left[z_k z_k^{*2} - \frac{1}{N} T_2^* z_k - \frac{1}{\tilde{T}_2} z_k^* \left(\tilde{T}_4 - \frac{1}{N} |T_2|^2 \right) \right]. \quad (\text{D.3})$$

If one sums the first equation over k , enforcing the trace condition, we find

$$C^* = \frac{18\lambda^2}{\tilde{T}_2^2 N} \tilde{T}_3^*, \quad (\text{D.4})$$

resulting in the root equation Eq. (5.16).

One may use the auxiliary function G to explore the second variation as well. Suppose we expand about an extremum \hat{z} of the form of Eq. (5.8). Writing $z = \hat{z} + \delta z$, then it can be shown that

$$\delta^2 G = \delta^2 V_F = \frac{1}{2} \sum_{i,j} \left[\frac{\partial^2 G}{\partial z_i \partial z_j} \Big|_{\hat{z}} \delta z_i \delta z_j + \frac{1}{2} \frac{\partial^2 G}{\partial z_i \partial z_j^*} \Big|_{\hat{z}} \delta z_i \delta z_j^* \right] + c.c., \quad (\text{D.5})$$

where the partial derivatives are at \hat{z}, \hat{z}^* , and the variations obey the constraint $\sum \delta z_k = 0$. In other words, one may treat the components z_k as independent and C as a constant in carrying out the derivatives, provided one enforces the constraint condition at the end⁷. Therefore, we simply need to calculate the matrix of second derivatives (or Hessian) of V_F :

$$\frac{\partial^2 G}{\partial z_i^* \partial z_j^*} = \frac{36\lambda^2}{N\Omega^2} \left[\delta_{ij} (Nz_j^2 - T_2) - a^2 z_i z_j - 2 \frac{\tilde{T}_3}{\Omega} (z_i + z_j) \right], \quad (\text{D.6})$$

$$\frac{\partial^2 G}{\partial z_i \partial z_j^*} = \frac{36\lambda^2}{N\Omega^2} \left[\delta_{ij} (2N|z_j|^2 - a^2\Omega) - 2z_i z_j^* - a^2 z_i^* z_j - \frac{2}{\Omega} (z_j \tilde{T}_3^* + z_i^* \tilde{T}_3) \right], \quad (\text{D.7})$$

where

$$a^2 \equiv \kappa^2 - \kappa_2^2, \quad \kappa^2 \equiv \frac{N\tilde{T}_4}{\Omega^2}, \quad \kappa_2 \equiv \frac{|T_2|}{\Omega}. \quad (\text{D.8})$$

One can show that $\kappa^2 \geq 1 \geq \kappa_2$, so that $a^2 \geq 0$. Some remarks are in order in how we arrived at these expressions. Since each component \hat{z}_k satisfies the root equation Eq. (5.16), wherever we encountered a cubic such as $z_j^2 z_j^*$, we replaced it with the corresponding linear terms using the root equation in the form

$$Nz^2 z^* = z a^2 \Omega + z^* T_2 + \tilde{T}_3. \quad (\text{D.9})$$

It is often convenient to decompose the second variation Eq. (D.5) in terms of real and imaginary parts, $\delta z_i \equiv \delta x_i + i\delta y_i$.

$$\begin{aligned} \delta^2 V_F &= \Re[G_{i\bar{j}} + G_{ij}] \delta x_i \delta x_j + 2\Im[G_{i\bar{j}} - G_{ij}] \delta x_i \delta y_j + \Re[G_{i\bar{j}} - G_{ij}] \delta y_i \delta y_j \\ &\equiv \frac{1}{2} (\delta x_i \ \delta y_i) \begin{pmatrix} A_{ij} & C_{ij} \\ C_{ji} & B_{ij} \end{pmatrix} \begin{pmatrix} \delta x_j \\ \delta y_j \end{pmatrix}, \end{aligned} \quad (\text{D.10})$$

where each matrix is obviously symmetric and real. However, we must recall that these variations are constrained by the traceless condition. Perhaps the easiest way to take that into account is to explicitly eliminate one of these coordinates, say,

$$\delta x_1 = - \sum_2^N \delta x_\alpha, \quad \delta y_1 = - \sum_2^N \delta y_\alpha. \quad (\text{D.11})$$

⁷This statement is true for nonlinear constraints as well.

Then the problem reduces to analysing the independent $(N-1)$ -dimensional variations

$$\frac{1}{2} (\delta x_\alpha \ \delta y_\alpha) \begin{pmatrix} \tilde{A}_{\alpha\beta} & \tilde{C}_{\alpha\beta} \\ \tilde{C}_{ji} & \tilde{B}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \delta x_\beta \\ \delta y_\beta \end{pmatrix}, \quad (\text{D.12})$$

where $\tilde{A}_{\alpha\beta} \equiv A_{\alpha\beta} + A_{11} - A_{1\beta} - A_{\alpha 1}$, and similarly for $\tilde{B}_{\alpha\beta}$ and $\tilde{C}_{\alpha\beta}$. For real extrema, $\tilde{C}_{\alpha\beta} = 0$, so the second variation obviously factors into the sum of separate variations of the real and imaginary parts.

Finally, for the case of interest, \hat{z} is real and of the form of Eq. (5.20). Taking $\hat{x} = \{2, 2, 2, -3, -3\}$ and $N = 5$, we find

$$\begin{aligned} A_{ij} &\propto \left[\frac{5}{3} \delta_{ij} (3\hat{x}_j^2 - 7) - \frac{7}{9} \hat{x}_i \hat{x}_j + \frac{4}{3} (\hat{x}_i + \hat{x}_j) \right], \\ B_{ij} &\propto \left[\frac{5}{3} \delta_{ij} (\hat{x}_j^2 + 5) - \frac{2}{3} \hat{x}_i \hat{x}_j \right]. \end{aligned} \quad (\text{D.13})$$

One may then easily compute the constrained variation Eq. (D.12) by eliminating one of the components of δx_k and δy_k . One then finds each matrix $\tilde{A}_{\alpha\beta}, \tilde{B}_{\alpha\beta}$ has one zero eigenvalue corresponding to displacements proportional to \hat{x} , plus 3 positive eigenvalues, showing that fluctuations in directions other than the flat direction are stable.

Of course, neglecting loop corrections to the potential, the magnitude of the field Φ is completely arbitrary, by classical scale invariance, so that $\phi\hat{x}$ is a good candidate for an inflationary field. Scale invariance will be broken at one-loop, and one would expect the scale ϕ to be determined via dimensional transmutation.

E V_F for Large ξ

Here we show that, in scale invariant models, $V_1 + V_2 \geq 0$ to leading order in ξ and that the correction terms are negative. Starting from Eqs. (5.1), (5.2), we may write

$$V \equiv V_1 + V_2 = \frac{1}{\Omega D} \left[\Omega D \left| \frac{\partial W}{\partial \phi^a} \right|^2 - |\Delta|^2 \right] \quad (\text{E.1})$$

with

$$\Omega = \frac{1}{M_P^2} \left[\sum \phi_a^* \phi^a - \frac{1}{2} \xi (c_{ab} \phi^a \phi^b + c.c.) \right] \quad (\text{E.2})$$

$$\Omega D = \xi^2 c_{ab} \phi^b c^{ac} \phi_c^* - \frac{\xi}{2} (c_{ab} \phi^a \phi^b + c.c.) \quad (\text{E.3})$$

$$\Delta = \xi \frac{\partial W}{\partial \phi^a} c^{ab} \phi_b^*. \quad (\text{E.4})$$

(Here, M_P is an arbitrary scale.) Inserting Eqs. (E.3), (E.4) into Eq. (E.1), we find for the quantity in brackets

$$\left[\xi^2 \left((c_{ab} \phi^b c^{ac} \phi_c^*) W_d W^d - |W_a c^{ab} \phi_b^*|^2 \right) - \frac{\xi}{2} (c_{ab} \phi^a \phi^b + c.c.) W_d W^d \right], \quad (\text{E.5})$$

where we have defined

$$W^a = \frac{\partial W^*}{\partial \phi_a^*}. \quad (\text{E.6})$$

The $O(\xi^2)$ terms are nonnegative by the Cauchy-Schwarz inequality. Further, it vanishes if and only if $W_a \propto \phi_a^*$ for the extremum values of the fields.

The factor in front, $1/\Omega D$, is positive and proportional to ξ^2 for large ξ . Therefore, V approaches a non-negative constant asymptotically:

$$V \rightarrow W_d W^d - \frac{|W_a c^{ab} \phi_b^*|^2}{c_{ab} \phi^b c^{ac} \phi_c^*}. \quad (\text{E.7})$$

Concerning the first correction terms of order $1/\xi$, they come from two sources: the second term in Eq. (E.5) and the second term in Eq. (E.3) in the factor in front. When combined, we find the leading correction to Eq. (E.7)

$$- \frac{1}{\xi} \frac{|W_a c^{ab} \phi_b^*|^2}{(c_{ab} \phi^b c^{ac} \phi_c^*)^2} \Re(c_{ab} \phi^a \phi^b). \quad (\text{E.8})$$

Under our assumptions, the factor $\Re(c_{ab} \phi^a \phi^b)$ is positive, so the $O(1/\xi)$ term above is negative. However, $V_F = 9V/\Omega^2$ in Einstein frame, so all such models have $V_F \rightarrow O(1/\xi^2)$ as $\xi \rightarrow \infty$. The leading term is

$$V_F \rightarrow \frac{9}{\xi^2 (\Re c_{ab} \phi^a \phi^b)^2} \left(W_d W^d - \frac{|W_a c^{ab} \phi_b^*|^2}{c_{ab} \phi^b c^{ac} \phi_c^*} \right). \quad (\text{E.9})$$

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